Dynamical Simulations of Probabilities

Michail Zak
Center for Space Microelectronics Technology
Jet Propulsion Laboratory
California Institute of Technology
Pasadena, CA 01109

Abstract

It has been demonstrated that classical probabilities, and in particular, probabilistic Turing machine, can be simulated by combining chaos and non- Lipschitz dynamics, without utilization of any man-made devices (such as random number generators). Self-organizing properties of systems coupling simulated and calculated probabilities and their link to quantum computations are discussed. Special attention was focused upon coupled stochastic processes, defined in terms of conditional probabilities, for which joint probability does not exist. Simulations of quantum probabilities are also discussed.

Classical dynamics is fully deterministic if initial conditions are known exactly. Otherwise in some non-linear systems, small initial errors may grow exponentially so that the system behavior attains stochastic-like features, and such a behavior is called chaotic. The discovery of chaos contributed in better understanding of irreversibility in dynamics, of evolution in nature, and in interpretation and modeling of complex phenomena in physics and biology. However, there is a class of phenomena which cannot be represented by chaos directly. This class includes so called discrete events dynamics where randomness appears as point events, i.e., there is a sequence of random occurrences at fixed or random times, but there is no additional component of uncertainty between these times. The simplest example of such a phenomenon

is a heartbeat dynamics which, in the first approximation, can be modeled by a sequence of pulses of equal heights and durations, but the durations of the pauses between these pulses are randomly distributed, Most processes of this type are associated with intellectual activities such as optimal behavior, decision making process, games, etc. In general, discrete events dynamics is characterized by a well-defined probabilistic structure of a piece wise-deterministic Markov chains, and it can be represented by probabilistic Turing machine. On the contrary, a probabilistic structure of chaos, and even the appearance of chaos at all, cannot be predicted based only upon the underlying model without actual numerical runs. (The last statement can be linked to the Richardson's (1968) proof that the theory of elementary functions in classical analysis is undecidable). But is there a "missing link" between chaos and discrete events dynamics? And if it is, can this link be simulated based only upon physical laws without exploiting any man-made devices such as random number generators? A positive answer to this question would make a fundamental contribution to the reductionists view on intrinsic unity of science that all natural phenomena are reducible to physical laws. However, in addition to this philosophical aspect, there is a computational advantage in exploiting simulated probabilities instead of calculated ones in the probabilistic Turing machine: as shown by R. Feynman (1982), the exponential complexity of algorithms in terms of calculated probabilities can be reduced to polynomial complexity in terms of simulated probabilities.

In this paper we demonstrate that the missing link between chaos and a discrete event process' can be represented by non-Lipschitz dynamics, (Zak, 1994, 1996.)

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In order to illustrate the basic concepts of non- Lipschitz dynamics. consider a rectilinear motion of a particle of unit mass driven by a non- Lipschitz force:

$$\dot{\upsilon} = v\upsilon^{1/3}\sin\omega t, \quad v = const, \quad [\upsilon] = \frac{m^{2/3}}{\sec^{5/3}} \tag{1}$$

$$\dot{x} = \upsilon \tag{2}$$

where υ and x are the particle velocity and position, respectively.

Subject to the zero initial condition

$$\upsilon = 0 \qquad \text{at} \qquad t = o \tag{3}$$

equation (1) has a singular solution

$$\upsilon = 0 \tag{4}$$

and a regular solution

$$\upsilon = \pm\left(\frac{4v}{3\omega}\sin^2\frac{\omega}{2}t\right)^{3/2} \tag{5}$$

These two solutions coexist at t = O, and this is possible because at this point the Lipschitz condition fails:

$$\left|\frac{\partial\dot{\upsilon}}{\partial\upsilon}\right|_{t\to 0} = \frac{1}{3}v\upsilon^{-2/3}\sin\omega t\Big|_{t\to 0} \to \infty \tag{6}$$

Since

$$\frac{\partial\dot{\upsilon}}{\partial\upsilon} \rangle o \qquad \text{at } |\upsilon| \neq (), \qquad t \rangle 0 \tag{7}$$

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the singular solution (4) is unstable, and the particle departs from rest following the solution (5). This solution has two (positive and negative) branches [since the power in (5) includes the square root], and each branch can be chosen with the probability p and (1 -p) respectively. It should be noticed that as a result of (5), the motion of the particle can be initiated by infinitesimal disturbances (such motion never can occur when the Lipschitz condition holds: an infinitesimal initial disturbance cannot become finite in finite time).

Strictly speaking, the solution (5) is valid only in the time interval

$$0 \leq t \leq \frac{2\pi}{\omega} \tag{8}$$

and at $t \leq 2\pi / \omega$ it coincides with the singular solution (4)

For $t) 2\pi / \omega$ equation (4) becomes unstable, and the motion repeats itself to the accuracy of the sign in equation (5).

Hence, the particle velocity v performs oscillations with respect to its zero value in such a way that the positive and negative branches of the solution (5) alternate randomly after each period equal to $2\pi / \omega$.

Turning to equation (2), one obtains the distance between two adjacent equilibrium position of the particle:

$$\Lambda x_i = x_i - x_{i-1} = \pm \int_0^{2\pi/\omega} \left(\frac{4v}{3\omega} \sin \frac{\omega}{2} t \right)^{3/2} dt = 64(3\omega)^{-5/2} v^{3/2} = \pm h \tag{9}$$

Thus, the equilibrium positions of the particle are

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$$x_0 = 0, \qquad x_1 = \pm h, \qquad x_2 = \pm h \pm h \dots \tag{10}$$

while the positive and negative signs randomly alternate with probabilities p and (1 -p), respectively.

Obviously, the particle performs an unrestricted random walk: after each time period

$$\pi = \frac{2\pi}{\omega} \tag{11}$$

it changes its value on $\pm h$ [see equation (10)].

The probability density f(x,t) is governed by the following difference equation:

$$j\text{-}(x, \ t + \tau) = pf(x - h, \ t) + (1 - p)f(x + h, t) \tag{12}$$

which represents a discrete version of the Fokker-Planck equation,
while

$$\int_{-\infty}^{\infty} f(x,t) \, dx = 1 \tag{13}$$

Several comments to the model (1) and its solution have to be made.

Firstly, the "viscous" force

$$F = - \nu \upsilon^{1/3} \tag{14}$$

includes static friction (see Eq. 6) which actually causes failure of the Lipschitz condition. These type of forces are well-known in theory of visco-plasticity (H.

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Ziegler, 1963). It should be noticed that the power $1/3$ can be replaced by any power of the type:

$$k = \frac{2n - 1}{2n + 1}, \quad n = 1, 2, \ldots \text{ etc} \tag{15}$$

with the same final result (1 2). In particular, by selecting large n, one can make k close to 1, so that the force (13) will be almost identical to its classical counterpart

$$F_c = - v\upsilon \tag{16}$$

everywhere excluding a small neighborhood of the equilibrium point $\upsilon = 0$, while at this point

$$\frac{dF}{d\upsilon} \to \infty, \quad \text{but} \quad \left| \frac{\partial F_c}{\partial \upsilon} \right| \to 0 \quad \text{at} \quad v \to 0 \tag{17}$$

Secondly, without the failure of the Lipschitz condition (6), the solution to Eq. (1) could not approach its equilibrium v = O in finite time, and therefore, the paradigm leading to random walk (12) would not be possible.

Finally, we have to discuss the infinitesimal disturbances mentioned in connection with the instability of the solutions (5) at υ =0. Actually the original equation should be written in the form:

$$\dot{\upsilon} = v\upsilon^{1/3} \sin\omega t + \varepsilon(t), \quad \varepsilon \to O \tag{18}$$

where $\varepsilon(t)$ represents a time series sampled from an underlying stochastic process representing infinitesimal disturbances. It should be emphasized that this process is not driving the solution of Eq. (1 8): it only triggers the mechanism of instability which controls the energy supply via the harmonic oscillations $\sin \omega t$.   As follows from Eq. (1 8), the function $\varepsilon(t)$ can be ignored when $\dot{\upsilon} = O$ or when $\dot{\upsilon} \neq O$, but the equation is stable, i.e. $\upsilon = \pi \omega, 2\pi \omega, \dots$ etc. However, it becomes significant during the instants of instability when $\dot{\upsilon} = 0$ at $t = 0, \pi/2\omega$ etc. Indeed, at these instants, the solution to Eq. (1) has a choice to be positive or negative if $\varepsilon = O$, (see Eq. (5)). However, with $\varepsilon \neq 0$,

$$sign\, x = sign\, \varepsilon \text{ at } t = 0,\ \pi/2\omega, \dots \text{etc} \qquad (19)$$

i.e., the sign of ε at the critical instances of time (19) uniquely defines the evolution of the dynamical system (18).   Thus, the dynamical system (18) transforms a stochastic process (via its sample $\varepsilon(t)$) into a binary time series which, in turn, generates a random-walk-paradigm (1 8). Actually the solution to Eq. (18) represents a statistical signature of the stochastic process ε.

Within the framework of dynamical formalism, the time series $\varepsilon(t)$ can be generated by a fully deterministic (but chaotic) dynamical system. The simplest of such system is the logistic map which plays a central role in population dynamics, chemical kinetics and many other fields. In its chaotic domain

$$y_{n+1} = 4y_n(1 - y_n),\, y_0 = 0.2 \qquad (20)$$

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the power spectrum for the solution is indistinguishable from a white noise. However, for the better match with Eq. (18), we will start with a continuous version of (20) represented by the following time-delay equation.

$$y(t + \tau) = 4y(t)[1 - y(t)], \tau = \frac{\pi}{2\omega} \tag{21}$$

$$y(t^*) = ().2, \quad --\frac{\pi}{4\omega} \ (t^* \ \langle \frac{\pi}{4\omega} \tag{22}$$

The solution to Eq. (21) at t=O, $\pi/2\omega$,... etc., coincides with the solution to Eq. (20), but due to the specially selected initial condition (22), the solution to Eq. (20) changes its values at $t = -\frac{\pi}{4\omega}, -*-, \ldots$ etc., so that at the points

t=o, $\pi/2\omega$,... , the sign of this solution is well-defined.

Now assume that

$$\varepsilon(t) = \varepsilon_0(y(t) - (),51), \quad \varepsilon_0 \ ((\ 1. \tag{23}$$

The subtraction from y(t) its mean value provides the condition

$$p = 1 - p = \frac{1}{2} \tag{24}$$

Indeed, for the first hundred points in (23),

$$Sign \ \varepsilon \ = \begin{array}{l} -++-+++--+--+------++++--+ \\ --++-+-+--++-----+-+----++-+ \\ +-++-+++--+++++++-+++++++---+ \\ +-----++----+-+---+---- \end{array} \tag{25}$$

has equal number of positive and negative values which are practically not correlated, Therefore, the statistical signature of the chaotic time series (23) is expressed by the solution to Eqs. (1 2), (13) at $p = \dfrac{1}{2}$ with the initial conditions

$$f(0,0) = 1, f(x,0) = O \text{ if } x \neq O \tag{26}$$

which is a symmetric unrestricted random walk:

$$f(x,t) = C_n^m 2^{-n}; \quad m = \frac{1}{2}(n+x); \quad n = \text{integer}\left(\frac{2\omega t}{\pi}\right) \tag{27}$$

Here the binomial coefficient should be interpreted as O whenever m is not an integer in the interval $[0, n]$, and n is the total number of steps.

The connection between the solution (26) and the solutions to the system (1 8), (21), (2) should be understood as follows. Suppose we solve the system (18), (21), (2) subject to the initial condition (22) with $v = 0$ and $x = 0 \text{ at } t = 0$.

Since Eq. (21) is supersensitive to inevitable errors in (22), the solution will form an ensemble of chaotic time series, and for any fixed instant of time this ensemble will have the corresponding probability distribution which coincides with (26). In other words, the probabilities described by Eq. (1 2), are simulated by the dynamical system (1 8), (21) and (2) without an explicit source of stochasticity (while the "hidden" source of stochasticity is in finite precision of the initial condition (22)).

Combining several dynamical systems of the type (18), (21), (2). and applying an appropriate change of variables, one can simulate a probabilistic Turing machine which transfers one state to another with a prescribed transitional probabilities. Non-Markovian properties of such a machine can be incorporated by introducing time-delay terms in Eq. (2).

$$\dot{x} = \text{`u}(/) + \alpha_1 v(t - \tau_0) + \alpha_2 v(t - 2\tau_0) + ...$$ (28)

However, there is a more interesting way to enhance the dynamical complexity of the system (1 8), 21), (2). Indeed, let us turn to Eq. (23) and introduce a feedback from Eq. (2) to Eq. (18) as following:

$$\varepsilon = \varepsilon_0(y - x), \; \varepsilon_0 \; ((1, \tilde{y} = y - 0. \; 51$$ (29)

Then the number of negative (positive) signs in the string *(25)* will prevail if x) O $(x$ (O) since the effective zero-crossing line moves down (up) away from the middle. Thus, when (x = O) at t =0, the system starts with an unrestricted random walk as described above, anti $|x|$ grows. However, this growth changes signs in Eq. (23) such that $\dot{x}$ (O if $_\mathbf{x}$) o, and $\dot{x}$) 0 if x (O. As a result of that

$$x_{\max} \leq y_{\max}, \quad \text{`rein} \geq y_{\min}$$ (30)

where $y_{\max}$ and $y_{\min}$ are the largest and the smallest values in the time series y (t), respectively. Hence, the dynamical system (1 8), (23), (2) simulates a restricted random walk with the boundaries (30) implemented by the dynamical feedback (29)', while the probability

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$$p(sign\varepsilon\rangle 0) = 0 \begin{cases} \text{if } x \geq y_{max} \\ 1 \text{ if } x \leq y_{min} \end{cases} \tag{31}$$

For the sake of qualitative discussion, assume that p change linearly between $x = y_{min}$ and $x = y_{max}$, i.e.,

$$P \begin{cases} 0 & \text{if } x > y_{max} \\ = \dfrac{y_{max} - x}{y_{max} - y_{min}} & \text{if } y_{min} \leq x \leq y_{max} \\ 1 & \text{if } x < y_{min} \end{cases} \tag{32}$$

(the actual function p(x) depends upon statistical properties of the underlying chaotic time series y(t). In particular, for the logistic map (20), small deviations from (32) take place only around the ends (i.e., when $x \cong y_{max}$ or $x \cong y_{min}$).

Then the simulated restricted random walk as a solution to Eqs. (12) and *(32.)*

Let us modify the feedback (29) as

$$\varepsilon = \varepsilon_0\left[y - \left(x^2 - x\right)\right] \tag{33}$$

Now when x=O at t=O, the system is unstable since

$$sgn\, x = sgn\, \dot{x}, \ -\infty \langle x \langle \ -\frac{1}{2}, \tag{34}$$

and the process is divided into two branches. The negative branch (with the probability 1/2) represent an unrestricted random walk ($x \rightarrow$ 00), while the

positive branch (with the same probability $\frac{1}{2}$) is eventually trapped within the basin of the attractor $x = 1$ since

$$sgn\ x\ =\ -sgn\ \dot{x},\ \frac{1}{2}\ \langle\ x\ \langle\ \infty \tag{35}$$

simulating a restricted random walk as those described above with the only difference that its center is shifted from $x = O$ to $x = 1$.

As a next step in complexity, introduce the information H associated with the random walk process described by Eqs. (12), (13):

$$H = -\int_{-\infty} f\log_2 f\,dx \tag{36}$$

and modify the feedback (29) as following:

$$\varepsilon = \varepsilon_0\ [y-\ x(1 + H)] \tag{37}$$

Following the same line of argumentation as those performed for the feedback (29), one concludes that the feedback (38) becomes active only if the process is out of the domain of the maximum information, and therefore, it is always attracted to this domain.

Since Eq. (31) is still valid, we will apply the approximation similar to (32):

$$p = \begin{cases} 0 \; if \; x\,(1+H) \geq y_{max} \\ \dfrac{y_{max} - x\,(1+H)}{y_{max} - y_{min}} \\ 1 \; if \; x\,(1+H) \leq y_{min} \end{cases} \tag{38}$$

in order to continue our qualitative analysis. It should be noticed that now p depends not only on x, but also on f, and that makes Eq. (12) nonlinear. In addition to that, the system (1 8), (2) and (37), which is simulating probabilities, is coupled with the system (12), (13) and (38) describing the evolution of calculated probabilities. Actually due to this coupling, the entire dynamical system attains such a self-organizing property as to maximize the information generated by the random walk.

The self-organizing properties of the system (18), (2), (37), (12),(13) and (38) mentioned above have a very interesting computational interpretation: they provide a mutual influence between different branches of probabilistic scenarios. Such an influence or interference, is exploited in hypothetical quantum computer (P. Shor, 1996) as a more powerful tool in a complexity theoretic sense, than classical probabilistic computations. However, in quantum computer, the interference is restricted to a linear unitary matrix transformation of probabilities (which is the only one allowed by quantum mechanics laws), while in the classical system (1 8), (2), (37) there is no such restriction; by choosing an appropriate probabilistic term in the feedback (37), we can provide an optimal interference. The price paid for such a property is the necessity to exploit the calculated probabilities (1 2), (13) and (38).

Finally consider the following two dimensional system:

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$$\dot{v}_{1=} v v_1^4 \sin \omega t + \varepsilon_o \left(\tilde{y} - x_2 \right), \quad \dot{x}_1 = v_1 \tag{39}$$

$$\dot{v}_2 = v v_2^4 \sin \omega t + \varepsilon_o \left(\tilde{y} - x_1 \right), \quad \dot{x}_2 = v_2, \tag{40}$$

with respect to variables x_1 and X_2

Eqs. (39) and (60) are coupled via the feedbacks, but their associated probability equations are not coupled:

$$f_1 \left(x_1, x_2, t + \tau \right) = p_1 \left(x_2 \right) f_1 \left(x_1 \cdot h, x_2, t \right) + \left[1 \cdot p_1 \left(x_2 \right) \right] f_1 \left(x_1 + h, x_2, t \right) \tag{41}$$

$$f_2 \left(x_1, x_2, t + \tau \right) = p_2 \left(x_1 \right) f_2 \left(x_1, x_2 - h, t \right) + \left[1 - p_2 \left(x_1 \right) \right] f_2 \left(x_1, x_2 + h, t \right) \tag{42}$$

where

$$p_1 = \begin{cases} O \text{ if } \bar{x}_2 > 1 \\ \tfrac{1}{2} \left(1 - \bar{x}_2 \right) if \left| \bar{x}_2 \right| \leq 1 \\ 1 \text{ if } \bar{x}_2 < 1 \end{cases}, \qquad P? \left| \begin{matrix} 0 \text{ if } \bar{x}, \geq 1 \\ \tfrac{1}{2} \left(1 - \bar{x}_1 \right) if \left| \bar{x}_1 \right| \leq 1 \\ 1 \text{ if } \bar{x}_1 < 1 \end{matrix} \right. \tag{43}$$

$$\left| y_{max} \right| = \left| y_{min} \right|, \quad \bar{x}_1 = \frac{x_1}{\left| y_{max} \right|}, \quad \bar{x}_2 = \frac{x_2}{\left| y_{max} \right|} \tag{44}$$

It should be noted that x_2 and x_1 enter as parameters into Eqs. (41), and (42) respectively. That is why $f_1 \left(x_1 \vert x_2 \right)$ and $f_2 \left(x_2 \vert x_1 \right)$ represent conditional

probability densities: f_1 describes the density of x_1 given x_2 and f_2 describes, the density of x_2 given x,.

The solution to Eqs. (41) and (42) subject to the initial conditions (26), and the condition (13) for a sufficiently small initial time interval t are:

$$f_1 = C_n^{m_1} \left(\frac{1 - \bar{x}_2}{2} \right)^{m_1} \left(\frac{\bar{x}_2 - 1}{2} \right)^{n - m_1} , \quad m_1 = \tfrac{1}{2} \left(n + \bar{x}_1 \right), n = \text{integer} \left(\frac{2\omega t}{\pi} \right), \qquad (45)$$

$$f_2 = C_n^{m_2} \left(\frac{1 - \bar{x}_1}{2} \right)^{m_2} \left(\frac{\bar{x}_1 - 1}{2} \right)^{n - m_2} , \qquad m_2 = \tfrac{1}{2} \left(n + \bar{x}_2 \right) \qquad (46)$$

Each of them represents a non-symmetric random walk before the reflections from the boundaries $|x_1| = 1, |x_2| = 1$ take place.

Now the following question can be asked: how to find an underlying joint probability density $\Phi(x_1 x_2)$? It turns out that this is a hard question even from a conceptual viewpoint. Indeed, the relationships between $f_1(x_1|x_2), f_2(x_2|x_1)$ and $\Phi(x_1, x_2)$ are the following:

$$\Phi(x_1, x_2) = f_1(x_1|x_2) \int_{-\infty}^{\infty} \Phi(z, x_2) dz = f_2(x_2|x_1) \int_{-\infty}^{\infty} \Phi(x_1, z) dz$$

whence

i.e.

$$\ell n \frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} = \ell n \int_{-\infty}^{\infty} \Phi(x_1,z)dz - \ell n \int_{-\infty}^{\infty} \Phi(z,x_2)dz$$

arid therefore

$$-\frac{\partial^2}{\partial x_1 \partial x_2} \cdot \ell n \frac{f_1(x_1|x_2)}{f_2(x_2|x_1)} = 0 \tag{47}$$

Thus the existence of the joint probability density $\Phi(x_1, x_2)$ requires that the conditional probability densities must satisfy the compatibility equation (47). But it is easily verifiable that the solutions (45) and (46) do not satisfy this equation, i.e., they are incompatible:

$$in\, k(\varphi_1, \varphi_2) = \frac{\partial^2}{\partial x_1 \partial x_2} \ell n \frac{f_1}{f_2} \neq 0 \tag{48}$$

At the same time, there is nothing wrong with these solutions since they describe two stochastic processes which can be implemented by dynamical simulations. Hence, the only conclusion which can be made is that the joint probability in this particular case does not exist! But how "particular" is this case? Based upon the degree of arbitrariness to which the feedbacks in the system (71)(72) can be "set up, it is obvious that the incompatibility of the conditional probabilities is a rule rather than an exception. In other words, there is a class of coupled stochastic processes for which joint probability does not

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exist, and therefore, they are inseparable i.e., there is no such transformation of variables which would break them down nto independent components.

The mathematical formalism exploited in all the previous discussions was based upon the relationships between the non-Lipschitz dynamical equations (I), (2) simulating random walk, and the discretized version of the Fokker-Planck equation (12) governing the evolution of the probability corresponding to this random walk.

A continuous version of Eq. (12), i.e., the Fokker-Planck equation, is obtained if

$$v \sim \omega^{4/3}, and \ \omega \rightarrow \infty \tag{49}$$

Indeed, then:

$$\tau = \frac{2\pi}{\omega} \rightarrow 0, \ h \sim \frac{1}{\omega^{1/2}} \rightarrow 0, \ \frac{h^2}{\tau} \rightarrow 2D = Const \tag{50}$$

and Eq. (12) reduces to the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2} \tag{51}$$

where f is the probability density

There is a mathematical similarity between the Fokker-Planck and the Schrödinger equations: e.g., (51) is formally equivalent to the Schrödinger equation with imaginary times:

$$t_{sch} = -i\,t,\ i = \sqrt{-1} \qquad (52)$$

Indeed, after replacing the probability density f in Eq. (51) by the probability amplitude ψ .

one arrives at the Schrödinger equation:

$$-i\frac{\partial \psi}{\partial t} = D\frac{\partial^2 \psi}{\partial x^2} \qquad , \qquad (53)$$

Continuing this analogy, one may ask: does there exist a dynamical system which simulates the Schrödinger equation (53) in the same way in which the dynamical system (1),(2) simulates the Fokker-Planck equation?

The formal mathematical answer to this question is very simple: yes, it does. Indeed, turning to Eqs. (I),(2) and introducing an imaginary time

$$t_* = it \qquad (54)$$

one obtains

$$\frac{dx}{dt_*} = v_*, \ \frac{(iv^*}{dt} = vv_*^{1/3}Sin\ \omega t_*,\ v - \ \omega^{1/3}, \omega \to \infty \qquad (55)$$

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Formally this system is identical to Eqs. (l),(2), and therefore, it describes a random walk whose probability is governed by the Fokker-Planck equation:

$$\frac{\partial f}{\partial t} = D\frac{\partial^2 f}{\partial x^2}$$

(56)

or, after returning to the real time t, and replacing f by ψ by the Schrödinger Equation (53).

Let us establish formal relationships between the parameters v and ω of the dynamical system (55), and the quantum characteristics of a particle. Identifying ω with the wave frequency of the particle, one obtains:

$$\omega = \frac{E}{\hbar}$$

(57)

where E is the particle energy, and $\hbar$ is the Planck constant,

The actual transition to the continuous limit from Eqs. (1 2), (13) to the Fokker-Planck equation (see Eqs. (49) and (50) is restricted by the uncertainty principle

$$2D = \lim\frac{(\Delta x)^2}{\Delta t} = \lim\Delta x\Delta v = \frac{\hbar}{m}$$

(58)

i.e.

$$D = \frac{\hbar}{2m}$$

(59)

where m is the mass of the particle.

Then, as follows from (9), (57) and (58):

$$v \sim \frac{E^{7/3}}{\hbar^{5/3} m^{2/3}}$$ (60)

Thus, Eqs. (57), (59) and (60) express the parameters of the dynamical system (55) and the corresponding Schrödinger equation via the physical characteristics of the particle.

Surprisingly, the mechanism of the instability of Eqs. (55) is explained much easier here than those in the classical case: it just follows from the uncertainty principle which rejects a possibility that initial conditions for both the position and the velocity of a particle are known exactly.

However, for the purpose of actual simulations of the Schrödinger equation (53), the dynamical system does not offer much (since it evolves in imaginary time), unless it can be given a meaningful physical interpretation. A mathematical formalism for such an interpretation can be borrowed from special theory of relativity in which physical events are mapped into a pseudo-eucledian space with real space coordinates and imaginary time. A much broader concept of conjugate complex time in physics in connection with transfinie Cantorian spaces, quantum field theories and the problem of irreversibility is discussed in recent papers by El Naschie (1995, 1996), El Naschie and 1. Prigogine (1996), El Naschie, Q. E. Rössler and G. Ord (1995).

However, the main question which we pose here is not in mathematics, but rather in physics: does the dynamical system (55) exist in real physical world?

The discovery of chaos in classical mechanics raised many questions among quantum physicists about a possibility that there is a deterministic microstructure behind the Schrödinger equation, and as a result of instability, this microstructure loses its determinism and "collapses" into probabilistic world in the same way in which deterministic Newtonian dynamics attains stochasticity due to chaos. Such speculations were encouraged by views expressed by A. Einstein who had never been comfortable with the probabilistic origin of quantum mechanics. From this viewpoint, the dynamical system (55) represents an alternative to this probabilistic origin: it is fully deterministic (since it does not include any random parameters); it is driven by instability triggered by uncertainties in initial conditions (in this context, the uncertainty principle in quantum mechanics plays the same role as the finite precision of initial conditions does in classical mechanics); and finally, the evolution of probability resulting from instabilities is described by the Schrödinger equation.

At this stage, we cannot prove (or disprove) existence of a deterministic origin of quantum mechanics. But we can make the following statement: if such a deterministic origin exists, its phenomenological structure is likely to be similar to those of Eqs. (55).

APPENDIX

Classical dynamics describes processes in which the future can be derived from the past, and in which the past can be traced from the future by time inversion, $t \rightarrow -t$. Because of such determinism and reversibility, classical dynamics becomes fully predictable, and therefore, cannot explain the emergence of new dynamical patterns in nature (as nonequilibrium thermodynamics can). This major flaw in classical dynamics has attracted the attention of many outstanding scientists (Gibbs, Planck, and Prigogine, among others).

Considering the governing equations of classical dynamics

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial R}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i} \qquad i = 1,2,\ldots,n \qquad \text{(AI)}$$

(where L is the Lagrangian, $q_i, \dot{q}_i$ are the generalized coordinates and velocity, and R is the dissipation function), we should recall that the structure of $R(\dot{q}_i, \ldots, \dot{q}_n)$ is not prescribed by Newton's laws: some additional assumptions must be made in order to define it. The "natural" assumption (which has been never challenged) is that these functions can be expanded in Taylor series with respect to equilibrium states $\dot{q}_i = 0$. Clearly, this requires the existence of the derivatives.

$$\left| \frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}j} \right| < \infty \quad at \ \dot{q}_i \rightarrow 0, \ q_j \rightarrow 0$$

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A departure from that condition is proposed by [Zak, 1992] where the following dissipation function is introduced.

$$R = \frac{1}{k+1} \sum_i \alpha_i \left| \sum_j \frac{\partial r_i}{\partial q_j} \dot{q}_j \right|^{k+1} \tag{A2}$$

in which

$$k = \frac{p}{p+2} < 1 \quad p \gg 1 \tag{A3}$$

where p is a large odd number. By selecting large p, we can make k close to 1, so that (A2) is almost identical to the classical assumption (when k=l) everywhere excluding a small neighborhood of the equilibrium point $\dot{q}_j = 0$; whereas, at that point

$$\left| \frac{\partial^2 R}{\partial \dot{q}_i \partial \dot{q}_j} \right| \to \infty \quad at \ \dot{q}_j \to 0 \tag{A4}$$

Thus, the Lipschitz condition is violated; the friction force $F_i = -(\partial R / \partial \dot{q}_i)$ grows at the equilibrium point, and then it gradually approaches its "classical" value. This effect can be interpreted as a mathematical representation of a jump from static to kinetic friction, where the dissipation force does not vanish with the velocity.

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It appears that this "small" difference between the friction forces at k=l and k < 1 leads to fundamental changes in Newtonian dynamics. In order to demonstrate this, we consider the relationship between the total energy E and the dissipation function R.

$$\frac{dE}{dt} = -\sum_i \dot{q}_i \frac{\partial R}{\partial \dot{q}_i} = -(k+1)R \tag{A5}$$

Within a small neighborhood of an equilibrium state (where the potential energy can be set to zero), the energy E and the dissipation function R have the respective orders

$$E \sim \dot{q}_i^2, R \sim \dot{q}_i^{k+1} \quad at \quad E \to 0 \tag{A6}$$

Hence, the asymptotic form of (A5) can be presented as

$$\frac{dE}{dt} = AE^{k+1/2} \quad at \ E \to 0, A = Const \tag{A7}$$

If A>0 and k<1, the equilibrium state E=0 is an attractor where the Lipschitz condition $\left(|d\dot{E}/dE| \to \infty \ at \ E \to 0\right)$ is violated. Such a terminal attractor is approached by the solution originated at $E = \Delta E_0 > 0$, in finite time, as follows.

$$t_0 = \int_{\Delta E_0}^0 \frac{dE}{AE^{(k+1)/2}} = \frac{2\Delta E_0^{(1-k)/2}}{(1-k)|A|} < \infty \tag{A8}$$

Clearly, this integral diverges in the classical case $k \geq 1$, where $t_0 \to \infty$. The motion described by (A7) has a singular solution $E \equiv 0$, and a regular solution

$$E = \left[\Lambda E_0^{(1-k)/2} + \frac{1}{2} A(1-k)t \right]^{-2/(1-k)}.$$
(A9)

In a finite time, the motion can reach the equilibrium and switch to the singular solution E=O, and this switch is irreversible.

The coefficient k can be found from experimental observations of the time t_0. In order to illustrate this, we consider a plane-incompressible flow, with a stream function ψ and the constitutive law

$$\sigma_{xy} = \mu_1 \left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right)^{k-1} \qquad \upsilon_x = \frac{\partial \psi}{\partial y}, = -\frac{\partial \psi}{\partial x}, \qquad k < 1$$
(Al O)

where $\sigma_{xy}, \upsilon_x,$ and υ_y are viscous stress and Cartesian projections of velocity. Based upon the relationship between the rate of change of the kinetic energy and the dissipation function, we obtain..

$$\frac{p}{2} \frac{\partial}{\partial t} \int_V \left[\left(\frac{\partial \psi}{\partial x} \right)^2 + \left(\frac{\partial \psi}{\partial y} \right)^2 \right] dx\, dy = -\mu_1 \int_V \left(\frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} \right)^{k+1} dx\, dy$$
(AII)

where p is density, μ_1 is viscosity, and V is te volume occupied by the fluid.

Suppose that $\psi(t,x,y)$ can be represented as a product $\psi = \overline{\psi}(t)\overline{\overline{\psi}}(x,y)$. then (Al 1) reduces to the ordinary differential equation with respect to $\varphi(t) = \overline{\psi}^2(t)$ as follows.

$$\dot{\Phi} = -\gamma v_1 \varphi^k \tag{Al 2}$$

and

$$\gamma = \frac{\int_v \left(\frac{\partial^2 \overline{\psi}}{\partial y^2} - \frac{\partial^2 \overline{\psi}}{\partial x^2}\right)^{k+1} dx\, dy}{\int_v \left[\left(\frac{\partial \overline{\psi}}{\partial x}\right)^2 + \left(\frac{\partial \overline{\psi}}{\partial y}\right)^2\right] dx\, dy} = Const, \quad v = \frac{\mu_1}{\rho}$$

Equation (A12) describes the damping of the fluid motion due to viscous stress (Al O). The equilibrium state represents a terminal attractor which is approached in a finite time:

$$t_0 = \frac{\varphi_0^{1-k}}{\gamma v_1 (1-k)} \quad \varphi_0 = \varphi(0) \tag{Al 3}$$

Equation (Al 3) allows one to evaluate k and v_1 from experimental measurements of t_0.

In conclusion, we stress again that all the new effects of terminal dynamics emerge within drastically diminishing neighborhoods of equilibrium states, which are the only domains where the governing equations are different from the classical models.

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